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EXACT ANALYTICAL SOLUTIONS OF FORCED CONVECTION FLOW IN A POROUS MEDIUM

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ABSTRACT

The forced convection problem in a fluid saturated porous medium is considered. For the self-similar boundary-layer flows past a plane or axisymmetric body with arbitrary shape embedded in this medium and having a power-law surface temperature distribution, analytical solutions are given. In the range $\lambda \geq 0$ of the temperature exponent, the analytical results show an excellent agreement with previous numerical findings. In the range $\lambda < -1/2$, the existence of a new class of unique solutions, while for $-1/2 < \lambda < 0$ the occurrence of multiple solutions is reported. The heat transfer characteristics and the physical meaning of all these forced convection boundary-layer flows are discussed in detail. © 2001 Elsevier Science Ltd

Introduction

Convective flow in porous media is one of the main topics of heat transfer, which has many practical applications. These include the utilization of geothermal energy, high performance insulation for buildings, the control of pollutant spread in groundwater, the design of nuclear reactors, compact heat exchangers, solar power collectors, food processing casting and welding of a manufacturing process, etc. Recent monographs by Nield and Bejan [1], Ingham and Pop [2] and Vafai [3] give a comprehensive summary of the work on the subject.

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The object of this Note is to present exact analytical solutions for the problem of forced convection flow over plane or axisymmetric bodies of arbitrary shape and a power-law surface temperature distribution which are embedded in a fluid-saturated porous medium. This problem has first been investigated by Nakayama and Koyama [4], Nakayama and Pop [5] and Nakayama [6,7]. In the range $\lambda \geq 0$ of the temperature exponent, the present analytical results compare excellently with the numerical results reported previously. In addition, in the range $\lambda < -1/2$ a new class of unique solutions and for $-1/2 < \lambda < 0$ a class of multiple solutions has been found. The paper discusses the heat transfer characteristics and the physical meaning of all these forced convection boundary-layer flows in detail.

Basic Equations

We consider a plane or axisymmetric body of arbitrary shape, which is embedded, in a fluid-saturated porous medium. The wall temperature of the heated body is $T_w(x)$, where x stands for the coordinate measured along the surface of the body. The external velocity of the fluid is $u_e(x)$ and its external temperature T_e is assumed to be constant. Nakayama and Pop [5] have shown that under the non- boundary-layer and Boussinesq approximation the basic equations of both the Darcy and Darcy mixed convection flows can be transformed to the following form:

$$f' = \frac{[(1+2Re^*)^2 + 4Gr^*\theta]^{1/2} - 1}{[(1+2Re^*)^2 + 4Gr^*]^{1/2} - 1} \tag{1}$$

$$\theta'' + \left(\frac{1}{2} - nI\right) f\theta' - nIf'\theta = Ix\left(f'\frac{\partial \theta}{\partial x} - \theta'\frac{\partial f}{\partial x}\right) \tag{2}$$

subject to the boundary conditions

$$\begin{aligned} &f = 0, \ \ \theta = 1 \ \ \text{on} \ \ \eta = 0 \\ &\theta \to 0 \ \ \text{as} \ \ \eta \to \infty \end{aligned} \tag{3}$$

Here f is the dimensionless stream function, θ is the dimensionless temperature, $n(x)$ and $I(x)$ are functions associated with the wall temperature distribution, Gr^* and Re^* are the modified local Grashof and Reynolds numbers for a porous medium and primes denote differentiation with respect to the similarly variable η.

If we now assume that $Re^* + Re^{*2} >> Gr^*$, which corresponds to the forced convection flow regime, Eq. (1) then gives

$$f' = 1 \tag{4}$$

In addition, we assume that the temperature difference $\Delta T_w(x) = T_w(x) - T_e$ varies as $\Delta T_w(x) \propto \xi^{\lambda}$ where λ is a constant and ξ is the transformed streamwise coordinate, see Nakayama and Koyama [4]. Then, we have

$$nI = \frac{\lambda}{1+2\lambda} \tag{5}$$

and Eq. (2) reduces to the following ordinary differential equation

$$\theta'' + \frac{1}{2(1+2\lambda)}\eta\theta' - \frac{\lambda}{1+2\lambda}\theta = 0 \tag{6}$$

subject to the boundary conditions

$$\theta(0) = 1 \tag{7a}$$

$$\theta \to 0 \quad \text{as} \quad \eta \to \infty \tag{7b}$$

Analytical Solutions

We shall now give some exact analytical solutions of Eq. (6) which were not been presented before.

1. Case $\lambda = 0$

Equation (6) now becomes

$$\theta'' + \frac{1}{2}\eta\theta' = 0 \tag{8}$$

with the solution

$$\theta = erfc\left(\frac{\eta}{2}\right) \tag{9}$$

if the boundary conditions (7) are used. Here erfc(...) denotes the complementary error function, see Abramowitz and Stegun [8].

The reduced wall heat flux is

$$\theta'(0) = -\frac{1}{\sqrt{\pi}} \cong -0.564189 \tag{10}$$

2. Case $\lambda = -1/2$

In this case, Eq. (6) reduces to

$$\eta\theta' + \theta = 0 \tag{11}$$

and it has the solution

$$\theta = \frac{const.}{\eta} \tag{12}$$

The solution (12) satisfies the boundary condition (7b) but it violates the condition (7a). This means that the boundary value problem governed by Eqs. (6) and (7) does not admit solutions on the η-scale if $\lambda = -1/2$. In order to obtain a solution for $\lambda = -1/2$, a scale change is required (see below).

3. The cases $\lambda \to \pm\infty$.

In both these cases, Eq. (6) reduces to

$$\theta'' - \frac{1}{2}\theta = 0 \tag{13}$$

and admits the solution

$$\theta = \exp\left(-\frac{\eta}{\sqrt{2}}\right) \qquad \text{with} \qquad \theta'(0) = -\frac{1}{\sqrt{2}} \tag{14a,b}$$

which obviously satisfies the boundary conditions (7).

4. The general case.

Using the new independent and dependent variables z and $Y(z,\lambda)$ defined by

$$z = \frac{\eta}{2\sqrt{\lambda + \frac{1}{2}}} \qquad \text{and} \qquad Y(z,\lambda) = \theta(\eta,\lambda)\, e^{+\frac{1}{4}z^2} \tag{15a,b}$$

one obtains from (6) the equation

$$\frac{d^2Y}{dz^2} - \left(\frac{1}{4}z^2 + 2\lambda + \frac{1}{2}\right)Y = 0 \tag{16}$$

of the parabolic cylinder functions, see Abramowitz and Stegun [8].

With the aid of the parabolic cylinder functions $Y(z,\lambda)$ the general solution of Eq. (6) can be expressed in terms of Kummer's confluent hypergeometric function $M(a,b,x)$ as, see Abramowitz and Stegun [8]

$$\theta(\eta,\lambda) = AM\left(-\lambda, \frac{1}{2}, -\frac{1}{2}z^2\right) + BzM\left(-\lambda + \frac{1}{2}, \frac{3}{2}, -\frac{1}{2}z^2\right) \tag{17}$$

where A and B are as yet arbitrary constants. The boundary condition (7a) implies $A = 1$, while B remains to be specified with the aid of (7b). On the other hand, having in view that the new independent variable (15a) is real only for $\lambda > -1/2$, while it becomes imaginary for $\lambda < -1/2$, it is convenient to write the general solution (17) in the form:

$$\theta(\eta,\lambda)=\begin{cases} M\left(|\lambda|,\frac{1}{2},+\frac{1}{2}\zeta^2\right)+B_1\zeta M\left(|\lambda|+\frac{1}{2},\frac{3}{2},+\frac{1}{2}\zeta^2\right) & \text{for } \lambda<-\frac{1}{2} \\ M\left(-\lambda,\frac{1}{2},-\frac{1}{2}z^2\right)+B_2 zM\left(-\lambda+\frac{1}{2},\frac{3}{2},-\frac{1}{2}z^2\right) & \text{for } \lambda>-\frac{1}{2} \end{cases} \tag{18}$$

where B_1 and B_2 are constants to be determined from (7b) and

$$\zeta=\frac{\eta}{2\sqrt{|\lambda|-\frac{1}{2}}} \tag{19}$$

is the real independent variable for $\lambda<-1/2$. Further, if we take into account that, see Abramowitz and Stegun [8],

$$M(a,b,x)\cong\begin{cases} \frac{\Gamma(b)}{\Gamma(a)}x^{a-b}e^x & \text{as } x\to+\infty \\ \frac{\Gamma(b)}{\Gamma(b-a)}(-x)^{-a} & \text{as } x\to-\infty \end{cases} \tag{20}$$

one obtains for θ the asymptotic behavior

$$\theta(\eta,\lambda)\underset{\eta\to\infty}{\cong}\begin{cases} \frac{\sqrt{\pi}}{2^{|\lambda|}}\left[\frac{\sqrt{2}}{\Gamma(|\lambda|)}+\frac{B_1}{\Gamma\left(|\lambda|+\frac{1}{2}\right)}\right]\zeta^{2|\lambda|-1}e^{+\frac{1}{2}\zeta^2} & \text{if } \lambda<-\frac{1}{2} \\ \frac{\sqrt{\pi}}{2^{\lambda}}\left[\frac{1}{\Gamma\left(\lambda+\frac{1}{2}\right)}+\frac{B_2}{\sqrt{2}\,\Gamma(\lambda+1)}\right]z^{2\lambda} & \text{if } \lambda>-\frac{1}{2} \end{cases} \tag{21}$$

where Γ denotes the Gamma function.

The asymptotic expression (21) shows that:

i) in the range $\lambda<-1/2$ the boundary condition (7b) can only be obtained if

$$B_1=-\sqrt{2}\,\frac{\Gamma\left(|\lambda|+\frac{1}{2}\right)}{\Gamma(|\lambda|)} \tag{22}$$

In this case, the solution is unique and has the expression

$$\theta(\eta,\lambda)=M\left(|\lambda|,\frac{1}{2},\frac{1}{2}\zeta^2\right)-\sqrt{2}\frac{\Gamma\left(|\lambda|+\frac{1}{2}\right)}{\Gamma(|\lambda|)}\zeta M\left(|\lambda|+\frac{1}{2},\frac{3}{2},\frac{1}{2}\zeta^2\right) \tag{23}$$

and the wall heat flux is given by

$$\theta'(0,\lambda)=-\frac{1}{\sqrt{2|\lambda|-1}}\frac{\Gamma\left(|\lambda|+\frac{1}{2}\right)}{\Gamma(|\lambda|)} \tag{24}$$

ii) in the range $-1/2<\lambda<0$ the boundary condition (7b) is satisfied for any value of $B_2 \equiv B$ and therefore (18) yields the multiple solutions:

$$\theta(\eta,\lambda)=M\left(-\lambda,\frac{1}{2},-\frac{1}{2}z^2\right)+BzM\left(-\lambda+\frac{1}{2},\frac{3}{2},-\frac{1}{2}z^2\right) \tag{25}$$

which decays algebraically like $\eta^{2\lambda}$ as $\eta\to\infty$. The wall heat flux of these multiple solutions is given by

$$\theta'(0,\lambda)=\frac{B}{\sqrt{2\lambda+1}},\qquad -\frac{1}{2}<\lambda<0 \tag{26}$$

iii) in the range $\lambda\geq 0$ the boundary condition (7b) can only be obtained if

$$B_2=-\sqrt{2}\frac{\Gamma(\lambda+1)}{\Gamma\left(\lambda+\frac{1}{2}\right)} \tag{27}$$

The solution is unique again and reads

$$\theta(\eta,\lambda)=M\left(-\lambda,\frac{1}{2},-\frac{1}{2}z^2\right)-\sqrt{2}\frac{\Gamma(\lambda+1)}{\Gamma\left(\lambda+\frac{1}{2}\right)}zM\left(-\lambda+\frac{1}{2},\frac{3}{2},-\frac{1}{2}z^2\right) \tag{28}$$

The wall heat flux is now given by

$$\theta'(0,\lambda)=-\frac{1}{\sqrt{2\lambda+1}}\frac{\Gamma(\lambda+1)}{\Gamma\left(\lambda+\frac{1}{2}\right)} \tag{29}$$

Discussion

The expressions (24) and (29) for the wall heat flux are negative for any $\lambda<-1/2$ and $\lambda\geq 0$, respectively. Therefore, the corresponding unique solutions (23) and (28) lead to the "usual" wall heat flux, directed from the wall to the fluid. All the temperature profiles show a monotonic decrease from 1 to zero as $\eta\to\infty$. In Table 1 a couple of values of $\theta'(0,\lambda)$ are given for different values of λ. In the range $\lambda \geq 0$ they show an excellent agreement with the results of Nakayama [7] obtained by numerically solving the boundary value problem (6)-(7). For both $\lambda \to \pm\infty$ one has $\theta'(0,\lambda) \to -1/\sqrt{2}$. In these limiting cases the corresponding solutions (23) and (28) reduce to the same elementary solution (14). It is easy to show that the limiting cases $\lambda \to \pm\infty$ of $\Delta T_w(x) \propto \xi^{\lambda}$ correspond on the x-scale to the exponential surface temperature distribution

$$\Delta T_w(x) = T_0 \exp\left(\gamma \int u_e r^{*^2} dx\right) \tag{30}$$

where T_0 and γ are constants. In the special case $\lambda = 0$, expression (27) reduces to (10) and from (26) one immediately recovers the elementary solution (9):

$$\theta(\eta,0) = 1 - \frac{\eta}{\sqrt{\pi}} M\left(\frac{1}{2},\frac{3}{2},-\frac{1}{4}\eta^2\right) = 1 - erf\left(\frac{\eta}{2}\right) = erfc\left(\frac{\eta}{2}\right) \tag{31}$$

As shown above, in the range $-1/2 < \lambda < 0$, the boundary condition (7b) is not able to specify the value of the constant $B_2 \equiv B$ and, therefore, it results the algebraically decaying multiple solutions (25). Obviously, for the special value of $B = B_2$ given by (27), the corresponding number of the multiple solutions (25) coincides precisely with the unique solution (28). In this case, an exponential one replaces the algebraic decay. However, in addition to this special value of B, there are an infinite number of other possible values of the constant B. They may be restricted to some extent by the physical requirement that $\theta(\eta;\lambda) > 0$ for any $0 \leq \eta < \infty$ and $-1/2 < \lambda < 0$. The necessary (but not sufficient) condition for this behavior of the temperature profile (25) is $B \geq B_2$, with B_2 being given by (27). The necessary and sufficient condition which is able to specify the constant B uniquely, emerges from the requirement that no heat flow exists at infinity, i.e. besides $\theta(\infty) = 0$ we also have

$$\theta'(\eta,\lambda) \to 0 \quad \text{as} \quad \eta \to \infty \tag{32}$$

and this is automatically satisfied for $\lambda < -1/2$ and $\lambda \geq 0$. By making use of the properties of Kummer's confluent hypergeometric function $M(a,b,x)$, it is easily to show that (32) is satisfied if and only if B coincides precisely with B_2 given by (27). In this way, (25) coincides with (28) and the latter represents the (exponentially decaying) solution of the problem (6)-(7) in the whole range $\lambda > -1/2$. The values of the wall heat flux calculated from (29) with $\lambda \in (-1/2, 0)$ coincide to a high accuracy with the values obtained numerically by Nakayama [7] for $\lambda = -0.1$ and -0.3.

Finally, the case $\lambda = -1/2$ deserves a special attention. As pointed out above, in this case the boundary value problem (6)-(7) does not admit solutions on the η -scale (see Eq.(12)). However, by whole range $\lambda > -1/2$. The values of the wall heat flux calculated from (29) with $\lambda \in (-1/2, 0)$ coincide with the values obtained numerically by Nakayama [7] for $\lambda = -0.1$ and -0.3 to a high accuracy, with the flow on the z -scale, such a solution does exist. It may be obtained as the special case $\lambda = -1/2$ and $B = 0$ of the multiple solutions (25) and reads:

$$\theta = M\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}z^2\right) = \exp\left(-\frac{1}{2}z^2\right) \tag{33}$$

This solution satisfies obviously the boundary conditions $\theta = 1$ on $z = 0$ and $\theta \to 0$ as $z \to \infty$ and leads to a vanishing wall heat flow: $d\theta / dz|_{z=0} = 0$.

We may conclude therefore that the forced convection problem considered in this paper admits (on the scale of the usual similarity variable η) exponentially decaying unique solutions for any real $\lambda \neq -1/2$. These solutions are available in the exact analytical form (23) and (28). In the range $-1/2 < \lambda < 0$ the unicity of the solutions requires in addition to the boundary condition (7b) that at infinity also the heat flux becomes vanishing (a requirement which for $\lambda < -1/2$ and $\lambda \geq 0$ is satisfied automatically). In all these cases a direct wall heat flow is obtained. Finally it is worth mentioning that, while in the case $\lambda > -1/2$ the numerical approach to the boundary value problem (6)-(7) is a standard matter, in the case $\lambda < -1/2$ it becomes a difficult task. This circumstance is connected to the fact that the asymptotic state $\theta = 0$ represents for $\lambda > -1/2$ a stable focus whereas for $\lambda < -1/2$ it becomes an unstable focus of Eq.(6). Therefore, the new analytic solutions (23) valid for $\lambda < -1/2$ plays an important role in this parameter range.

TABLE 1

Values of $-\theta'(\eta, \lambda)$

λ	$-\theta'(\eta, \lambda \geq 0)$ Numerical results Nakayama [7]	$-\theta'(\eta, \lambda \geq 0)$ Present Results Eq.(27)	λ	$-\theta'(\eta, \lambda < -1/2)$ Present Results Eq.(27)
0	0.564	0.564189	-0.55	1.90486
0.1	0.583	0.583176	-0.60	1.42848
0.2	0.598	0.597813	-0.65	1.23014
1/3	0.613	0.612781	-0.70	1.11841
0.5	0.627	0.626657	-0.75	1.04605
0.8	0.644	0.643608	-0.80	0.99519
1.0	0.651	0.651470	-1.0	0.886227
1.5	0.665	0.664670	-3.0	0.743124
2.0	0.673	0.672835	-5.0	0.726983

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