

The problem you want to solve is, find ϕ such that:

$$\left\{ \begin{array}{l} -\Delta\phi = 0, \text{ in } M \text{ with,} \\ \nabla\phi \cdot \mathbf{n} = f \text{ on } M_{xmin}, \\ \nabla\phi \cdot \mathbf{n} = -f \text{ on } M_{xmax}, \\ \nabla\phi \cdot \mathbf{n} = 0 \text{ on } M_{ymin} \cup M_{ymax} \cup M_{zmin}, \text{ and} \\ \alpha\phi + \nabla\phi \cdot \mathbf{n} = 0 \text{ on } M_{zmax}, \end{array} \right.$$

where \mathbf{n} is the exterior normal on the boundaries of M .

In order to “simplify” the reading of the establishment of this variational formula, I will deliberately omit the functional spaces needed here (namely Sobolev spaces). If you need more information about this, check P-G Ciarlet’s book on finite elements for instance.

We start considering the PDE defined in M . Here, we have

$$-\Delta\phi = 0.$$

Thus, for any (test) function ψ , we also have

$$\forall\psi, \quad -\Delta\phi \times \psi = 0.$$

which can be written also

$$\forall\psi, \quad \int_M -\Delta\phi \times \psi = 0.$$

Using integration by part (Green formula here) we will have the boundary conditions arising:

$$\forall\psi, \quad \int_M \nabla\phi \cdot \nabla\psi - \int_{\partial M} \nabla\phi \cdot \mathbf{n}\psi = 0.$$

Now we will replace the boundary conditions within $\int_{\partial M} \nabla\phi \cdot \mathbf{n}\psi$. We have

$$\begin{aligned} \int_{M_{xmin}} \nabla\phi \cdot \mathbf{n}\psi &= \int_{M_{xmin}} f\psi, \\ \int_{M_{xmax}} \nabla\phi \cdot \mathbf{n}\psi &= -\int_{M_{xax}} f\psi, \\ \int_{M_{ymin} \cup M_{ymax} \cup M_{zmin}} \nabla\phi \cdot \mathbf{n}\psi &= 0 \\ \int_{M_{zmax}} \nabla\phi \cdot \mathbf{n}\psi &= -\int_{M_{zmax}} \alpha\phi\psi. \end{aligned}$$

So the weak form is

$$\forall\psi, \quad \int_M \nabla\phi \cdot \nabla\psi + \int_{M_{zmax}} \alpha\phi\psi = \int_{M_{xmin}} f\psi - \int_{M_{xax}} f\psi.$$