In the event that $n - m = 1$, there are at most two distinct extreme points, and they are automatically adjacent since the line joining them is the entire convex set of feasible solutions.

The above paragraphs have demonstrated that two basic feasible solutions which differ from each other only in that one basic vector has been changed to convert one solution to the other, are either adjacent extreme points or the two solutions are degenerate and are merely different representations of the same extreme point. Hence, when we move from one tableau of the simplex method to the next, we either remain at the same extreme point ($\theta_{\max} = 0$) or move to an adjacent extreme point ($\theta_{\max} > 0$).

If in (5–12) $y_{ik} \leq 0$ for all i, then (5–12) is a feasible solution for any $\theta \geq 0$. In this case, we can also show that the set of feasible solutions (5–12) is an edge of the convex set. The proof is merely a reproduction of the above development. However, the edge does not lead to another extreme point, because θ can be made arbitrarily large without driving any variable to a negative value. The edge goes out to infinity instead. Since either all $y_{ik} \leq 0$ or at least one $y_{ik} > 0$, we see that an edge emanating from a given extreme point either leads to another extreme point or extends to infinity.

Finally, we wish to note that the matrix $\mathbf{A} = (\mathbf{a}_1, \ldots, \mathbf{a}_n)$ performs a linear transformation on the n-dimensional solutions space and takes it into the m-dimensional requirements space. In particular, the convex set of feasible solutions in the solutions space is taken into the single point $\mathbf{b}$ in the requirements space.

5–7 Determination of all optimal solutions. We have shown that if k different basic feasible solutions to a linear programming problem are optimal, any convex combination of these basic solutions is also an optimal solution. The simplex procedure, as we have discussed it, stops once an optimal basic feasible solution has been obtained. It is seldom that any effort is made to find alternative optima. In fact, most computer codes supply a single optimum and make no provision for determining other optimal basic feasible solutions (if there are any). Sometimes, useful information can be obtained from the knowledge of all optimal basic feasible solutions. Hence, it is desirable to show how they can be found. However, there should be no economic reason for preferring one optimal basic feasible solution to another since all optimal solutions should be equally good. If this is not the case, then incorrect prices were assigned to the activity vectors during the formulation of the problem.

The final simplex tableau is the starting point for finding all other optimal basic solutions which may exist. If the optimal solution represented by the last tableau is not degenerate, and if $z_j - c_j > 0$ for each $\mathbf{a}_j$ not in the basis, then the optimal basic feasible solution is unique. No vector can be inserted into the basis without decreasing the value of the objective function.

When $z_j - c_j = 0$ for one or more $\mathbf{a}_j$ not in the basis, any such vector $\mathbf{a}_j$ can be inserted to yield a different optimal solution if $y_{ij} > 0$ for at least one i and min (x_{Bi}/y_{ij}), $y_{ij} > 0$, is positive. If $\mathbf{a}_j$ enters at a zero level, we do not obtain a different solution; the result is only a different representation of the same degenerate extreme point. If $y_{ij} \leq 0$ for all i, then $\mathbf{a}_j$ can be inserted to give a set of optimal solutions containing at least two variables which can be made arbitrarily large. It is also true that if the optimal solution is degenerate, then any vector $\mathbf{a}_j$ for which $y_{ij} \neq 0$ for any i corresponding to an $x_{Bi} = 0$ can be inserted into the basis, and a new representation of the same degenerate extreme point will be obtained. This can be done even if $z_j - c_j > 0$ since $\mathbf{a}_j$ enters at a zero level.

The above paragraph suggests the procedure for finding all optimal basic feasible solutions. Starting from the final tableau, which contains an optimal solution to the problem, we construct a new set of tableaux, each new tableau differing from the final tableau only in that one vector in the basis is changed. For insertion into the basis, we consider the vectors $\mathbf{a}_j$ with $z_j - c_j = 0$ or vectors which can enter at a zero level (even if $z_j - c_j > 0$). If $\mathbf{a}_j$ enters at a positive level, we obtain an alternative optimal basic solution. When $\mathbf{a}_j$ enters at a zero level, we do not obtain a different solution. However, we construct these tableaux anyway, because in the subsequent steps they may lead to new optimal basic solutions.

We repeat the same procedure with each of the new tableaux, and obtain some other optimal solutions which may or may not be optimal solutions different from those obtained in the first step. This is continued with each set of new tableaux until it is no longer possible to find any optimal basic solutions different from those already obtained. It is desirable to keep a record of all optimal basic solutions to prevent the computation of new tableaux which only yield an optimal solution that has already been determined. In schematic form, the process is represented by a treelike structure, as shown for a hypothetical case in Fig. 5-7. The basic solutions corresponding to different extreme points have different letters, and different degenerate basic solutions corresponding to the same extreme point have different subscripts on the letters. In our example, there are six

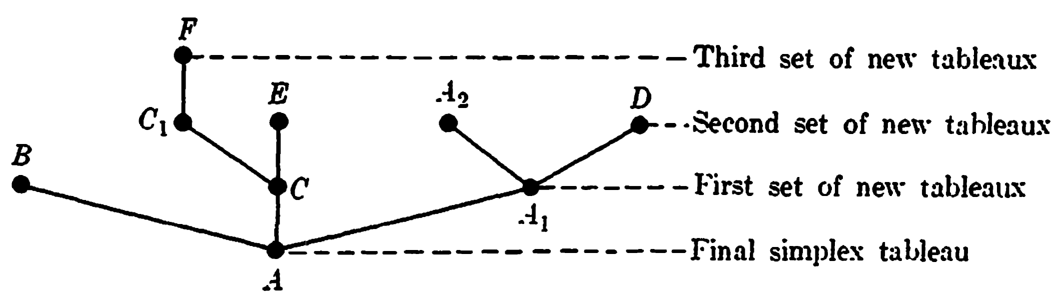

FIGURE 5-7

different optimal basic solutions (different extreme points) A, B, C, D, E, F. After constructing a sufficient number of tableaux, we only repeat solutions already obtained.

Clearly, if a fair number of optimal solutions exists, a good deal of work could be involved in determining all of them, since each change of basis requires the computation of a whole new tableau.

If desired, the second best and third best, etc., solutions to the problem can be found. To find the second best solution, for example, compute

$$\min_j \frac{x_{Br}}{y_{rj}} (z_j - c_j), \qquad z_j - c_j > 0, \qquad y_{rj} > 0, \qquad x_{Br} > 0. \quad (5\text{–}16)$$

This minimum is determined for each optimal tableau. Then the minimum of all these minima is found in order to obtain the smallest possible decrease in z.

Let us now consider the problem whose final tableau is given in Table 4–10. To the accuracy of the computations (and this is really all that can ever be said in numerical calculations where only a fixed number of digits is retained) there is a vector $\mathbf{a}_7$ (not in the basis) with $z_7 - c_7 = 0$. Thus the optimal solution is not unique. We know that there is one other basis that will yield the same optimal value of the objective function. Indeed, we can see immediately that there are only two basic optimal solutions. This follows since, after a vector with $z_k - c_k = 0$ has been inserted into the basis, $\hat{z}_j - c_j = z_j - c_j$, and the $(z_j - c_j)$-row in the new tableau with $\mathbf{a}_7$ in the basis will be the same as in Table 4–10. There is no vector other than $\mathbf{a}_7$ with $z_j - c_j = 0$. Furthermore, the optimal solutions are not degenerate, and hence a vector with $z_j - c_j > 0$ cannot be inserted.

We can easily compute the new optimal basic solution. Note that $\mathbf{a}_7$ replaces $\mathbf{a}_1$. Hence

$$x_{B1} = 16.91 - \frac{0.2727}{0.1818}(7.273) = 6.00 = x_6,$$

$$x_{B2} = \frac{7.273}{0.1818} = 40.0 = x_7,$$

$$x_{B3} = 6.364 + \frac{0.09091}{0.1818}(7.273) = 10.0 = x_3.$$

Of course, any convex combination of these two basic optimal solutions will also be an optimal solution.

5–8 Unrestricted variables. Thus far our discussion of linear programming has always been based on the assumption that the variables x_j are restricted to be non-negative. On occasions, one encounters problems of the linear programming type in which some or all of the variables can have any sign. Variables which can be positive, negative, or zero are called